A concept approach to input/output logic

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Abstract

This paper provides a semantics for input/input output logic based on formal concept analysis. The central result shows that an input/output logic axiomatised by a relation R is the same as the logic induced by deriving pairs from the concept lattice generated by R using a $\wedge$- and $\vee$-classical Scott consequence relation. This correspondence offers powerful analytical techniques for classifying, visualising and analysing input/output relations, revealing implicit hierarchical structure and/or natural clusterings and dependencies. The application of all formal developments are illustrated by a worked example towards the end.

Key words: Input/output logic, formal concept analysis, production inference, generation relations

1. Introduction

Input/output logic is a branch of conditional logic, broadly conceived, whose distinguishing feature is that it does not make any assumptions about the ultimate nature of the relation that holds between a set of conditions and its consequences. That may not sound like much, but it makes a real difference, both formally and philosophically.

Philosophically, it goes against a habit of logicians of assimilating all kinds of connections between a condition and a consequence to the inference paradigm: causality becomes causal *inference*, by which is meant the drawing of conclusions about a causal connection based on the conditions for the occurrence of an effect. The study of sets of norms becomes the study of normative *reasoning*, by which is usually meant the application of a practical

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syllogism or the drawing of a conclusion about the optimality of some state of affairs.

Although this method of investigation is sometimes natural, it is certainly not inevitable, nor is it always the most direct strategy available. After all, studying causality by way of our reasoning about it is a somewhat roundabout way of approaching the object. The causal relation itself is not an inference relation, strictly speaking, it is a relation between *things* or natural phenomena. At best, co-variation of causes and effects correspond to conditionals only in a derivative sense therefore. Analogical remarks apply to sets of norms: a norm is not primarily a conditional, it is a stipulation that holds by decree.

Input/output logic gives expression to this latter way of looking at things. A correlation between a condition and a consequence is seen as just an element in an ordinary binary relation between states of affairs as described by formulae. The ultimate nature and origin of this relation is left open, which just means that input/output logic does not foreclose any interpretational options.

On a formal level, the thrust of this general stance—which is justified in more detail in [11]—is to shift the emphasis from a theory formulated in terms of the behaviour of object-language connectives to a theory formulated in terms of the behaviour of sets and relations. Its methodological significance consists in the fact that it allots to philosophical logic a parcel in a wider mathematical landscape where logic is naturally tangent upon e.g. lattice theory and universal algebra.

The present discussion proceeds subject to this general conception to study the particular tangiental point which exists between input/output logic and the branch of lattice theory called formal concept analysis (FCA for short). The basic idea is this: formal concept lattices offer powerful, well-studied, analytical techniques for classifying, visualising and analysing binary relations, revealing implicit hierarchical structure and/or natural clusterings and dependencies between the objects of the relation [5]. Since the set of axioms in any given input/output logic is just a binary relation between formulae, it ought to be possible to apply results from FCA to the study of forms of conditionality that are not naturally assimilated to the model based on inference relations and/or conditionals—e.g. to sets of norms (an example that will be used for illustration purposes throughout the paper). This idea was first proposed in [20], but without being developed in much detail.

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The principal research question of the present paper is the following: given a binary relation G over sets of formulae, what is the relation between the concept lattice induced by G on the one hand and the input/output logic *axiomatised* by G on the other? The paper does not answer this question generally, but limits itself to a couple of more specific results summarised below:

First, there is a faithful embedding into FCA of the system of input/output logic called *basic output* in the nomenclature of Makinson and van der Torre. More precisely, we have that A) the concept lattice induced by the relation G is equivalent (in a sense to be made precise) to G itself modulo basic output, and B) there is a rule for evaluating any pair of formulae (a, b) against the concept lattice induced by G that answers yes iff (a, b) is in system of basic output axiomatised by G. Stated differently, for any given system of basic output, the concept lattice induced by the axioms of that system can A) itself be turned into a set of axioms for the system, and B) constitutes a semantic structure for it.

The second result concerns the system of input/output logic that Makinson and van der Torre call *basic reusable output*, which is the system that results from adding a rule of cumulative transitivity to the set of rules for basic output. The theme is now varied a bit: whereas it would be convenient to try and extend the results for basic output in the straightforward manner and ask whether the concept lattice induced by G can be turned into a set of axioms for the system of basic *reusable* output having G as a set of axioms, this would not be a very interesting exercise given the goals of the present paper. The central concern of the present paper is to bring input/output logic into the ambit of formal concept analysis, more specifically to make it possible to draw on lattice theoretic techniques in the analysis of a given input/output system. Obviously, this requires information to be encoded in the concept lattice. Yet, if one were to extend the results for basic output in the straightforward manner, then the set of generators/axioms of the logic would remain the same and so would the corresponding concept lattice. In other words, the concept lattice would not reflect the change of logic.

Instead therefore, the strategy that chosen is to encode pairs whose derivation is licensed by the rule of cumulative transitivity into the concept lattice itself by extending the relation G to a larger relation G^+ that still induces a finite lattice. This process is referred to as *saturating* the axioms—or simply as saturation. It is proved that saturation yields an axiomatisation for a system of basic reusable output analogous to A) above, although the procedure

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is not exhaustive enough to supply a *non-trivial* semantics in the sense of B).

This notwithstanding, it is argued that saturation is instructive and helpful in precisely the way it was hoped that it would be, namely as offering a lattice theoretic basis for exploration of a transitive input/output logic. This claim is substantiated by working through an example towards the end of the paper, where it is suggested that the modular structure of a corpus of norms may fruitfully be analysed along lattice theoretic lines.

2. Preliminaries

Notation. Let L be a language containing a countable set of elementary letters. The zero-ary *falsum* $\bot$ is counted among the connectives of L, and $\top$ is defined as $\neg\bot$. Lower case letters $a, b, c...$ range over formulae of L, upper case letters $A, B, C, \ldots$ range over sets of formulae, and W - Z over arbitrary sets. As usual $\wedge$ and $\vee$ denote classical conjunction and disjunction, respectively, but will do dual service as infima and suprema in a lattice. That is, when A is a finite set of formulae then $\bigwedge A$ is taken to denote the conjunction and $\bigvee A$ the disjunction of its finitely many elements, whilst if X is an unspecified set then $\bigwedge X$ is meet and $\bigvee X$ is join. The upper case letter G ranges over $L \times L$, that is, G denotes a binary relation over sentences in L. For arbitrary relations of any arity the letter R will be used. For any relation R, R^i denotes the projection of R onto its i-th coordinate, whereas $r, s, \ldots$ range over elements of R. We adopt the convention of writing r^i instead of $\{r\}^i$.

2.1. Generalized consequence relations

The developments that follow assume a background sentential logic, typically classical logic, which it will be convenient to present in terms of a generalized consequence relation $\vdash \subseteq 2^L \times 2^L$ on a language L satisfying the following three conditions:

Reflexivity:	$a \vdash a$
Thinning:	If $A \vdash B$ then $A', A \vdash B, B'$
Cut:	If $A, a \vdash B$ and $A \vdash a, B$ then $A \vdash B$

Special care is required when dealing with the empty set, which is equivalent to the *falsum* on the right of the turnstile $\vdash$ and to the *verum* on the left of it. That is, $A \vdash \emptyset$ iff $A \vdash \bot$ and $\emptyset \vdash B$ iff $\top \vdash B$.

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It is common to interpret the comma on the left of the turnstile as conjunction and the comma on the right as disjunction, giving the following informal reading of $A, B \vdash C, D$: if the conjunction of sentences in the union of A and B is true then necessarily the disjunction of sentences in the union of C and D is true as well. It is in general dangerous to understand this in terms of *classical* conjunction and disjunction, since the meaning and behaviour of the comma depends on the relation $\vdash$. However, it is a reading that is encouraged for the purposes of the present paper, since consequence relations will henceforth be assumed to be $\wedge$-classical and $\vee$-classical (cf. [7, chp. 1.1.]):

Definition 2.1. *For arbitrarily chosen a and b*

1. $\vdash$ *is* $\vee$***-classical*** *iff* $a \vee b \vdash a, b$, $a \vdash a \vee b$ *and* $b \vdash a \vee b$.
2. $\vdash$ *is* $\wedge$***-classical*** *iff* $a, b \vdash a \wedge b$, $a \wedge b \vdash a$ *and* $a \wedge b \vdash b$.

2.2. Input/output logic

The theory of input/output logic was first developed in Makinson and van der Torre [13, 14, 15], where four systems are singled out for special attention: simple-minded output, basic output (making intelligent use of disjunctive inputs), simple-minded reusable output (in which outputs may be recycled as inputs), and basic reusable output [13]. These are defined semantically and characterised by different constellations of derivation rules taken from the following set:

$$CT \ \frac{(a,b),(a \wedge b, c)}{(a,c)} \qquad AND \ \frac{(a,b),(a,c)}{(a, b \wedge c)} \qquad WO \ \frac{(a,b)}{(a,c)} \ \ \textit{if}\ b \vdash c$$

$$OR \ \frac{(c,b),(a,b)}{(a \vee c, b)} \qquad SI \ \frac{(c,b)}{(a,b)} \ \ \textit{if}\ a \vdash c$$

Different choices of rules give different relations of derivability $\Vdash \subseteq 2^{L \times L} \times L$, where the concept of derivability is defined as follows:

Definition 2.2. *A* ***rule*** *R of arity $n \geq 0$ is an $n+1$-ary relation over the set $L \times L$ of pairs of formulae in the language L. For any sequence of pairs $((a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})) \in R$ the premises of the rule is the subsequence $(a_1, b_1), \ldots, (a_n, b_n)$ and (a_{n+1}, b_{n+1}) is its conclusion. A* ***derivation***

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of a pair (a, b) from a set of input/output axioms $G \subseteq L \times L$ by means of a set of rules X is understood to be a tree with (a, b) at the root, each non-leaf node related to its immediate parents by the inverse of a rule in X, and each leaf node an element of G or identical to $(\top, \top)$.

Note that definition 2.2 hard-wires the axiom $(\top, \top)$ into the very concept of derivability from G irrespective of whether or not G contains it. Anomalous, but harmless, this axiom reflects the original input/output semantics where $(\top, \top)$ follows semantically from any G modulo any input/output operator. The concept semantics developed in the section 3 does not have this feature, and the pair $(\top, \top)$ will eventually drop out of view.

Table 1 summarizes the nomenclature of [13]. Here, $\Vdash_n$ denotes derivability according to definition 2.2 for different choices of rule set identified by the subscript n:

Rule set	$\Vdash_n$	**Name**
WO, SO, AND	$\Vdash_1$	*Simple-minded output*
WO, SO, AND, OR	$\Vdash_2$	*Basic output*
WO, SO, AND, CT	$\Vdash_3$	*Reusable output*
WO, SO, AND, OR, CT	$\Vdash_4$	*Basic reusable output*

Table 1: Input/output logics.

Each $\Vdash_n$ is a Tarski consequence relation, that is, it satisfies the following conditions of *r*eflexivity, *m*onotony and *c*umulative *t*ransitivity:

R: $g \Vdash_n g$ any $g \in G$
M: If $G \Vdash_n g$ then $G, G' \Vdash_n g$
CT: If $G \Vdash_n g$ for each $g \in G'$ and $G, G' \Vdash_n g$ then $G' \Vdash_n g$

Each relation $\Vdash_n$ may also be seen as an operation taking a relation G to a larger relation $C_n(G) =_{df} \{(a, b) : G \Vdash_n (a, b)\}$. As so defined C_n is a closure operator satisfying *in*clusion, *m*onotony and *id*empotence:

In: $G \subseteq C_n(G)$
M: If $G \subseteq G'$ then $C_n(G) \subseteq C_n(G')$
Id: $C_n(G) = C_n(C_n(G))$

The two representations will be used interchangeably.

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Note that in the present paper there are now two kinds of consequence relation in play: the relation $\Vdash_n$ between $2^{L\times L}$ and L, and the relation $\vdash$ between 2^L and 2^L. The former is a Tarski consequence relation whereas the latter is a generalized consequence relation, aka. Scott consequence relation. The two should not be conflated.

2.3. Formal concept analysis

Formal concept analysis is a mathematical theory of data analysis based on lattice theory. It offers a principled way of deriving an implicit classificatory hierarchy from a dataset, grouping objects by the properties they satisfy. The taxons of such a hierarchy are called *formal concepts* which are considered to be determined by their *extent* and *intent*: the extent consists of all objects belonging to the concept, while the intent is the collection of all attributes shared by the objects [4, p. 65]. The relation between a set of objects and a set of attributes is called a context:

Definition 2.3. *A* ***context*** *is a triple* (W, X, R) *where* W *and* X *are sets and* $R \subseteq W \times X$*. The elements of* W *and* X *are called* ***objects*** *and* ***attributes*** *respectively.*

Obviously, any binary relation R can be considered a context with $W = R^1$ and $X = R^2$. Now, for $Y \subseteq W$ and $Z \subseteq X$, define

Definition 2.4.

1. $Y^{\triangleright} =_{df} \{x \in X : (\forall w \in Y)(w, x) \in R\}$
2. $Z^{\triangleleft} =_{df} \{w \in W : (\forall x \in Z)(w, x) \in R\}$

so $Y^{\triangleright}$ is the set of attributes common to all the objects in Y and $Z^{\triangleleft}$ is the set of objects possessing all the attributes in Z [4, p. 67]. The pair $(^{\triangleleft}, ^{\triangleright})$ forms a Galois connection between 2^Y and 2^Z in which $^{\triangleright}$ is the lower- and $^{\triangleleft}$ the upper adjoint:

Theorem 2.5. *If* (W, X, R) *is a context, and if* $Y_1, Y_2, Y_3 \subseteq W$ *are extents and* $Z_1, Z_2, Z_3 \subseteq X$ *are intents, then*

1) $Y_1 \subseteq Y_2 \Rightarrow Y_2^{\triangleright} \subseteq Y_1^{\triangleright}$	*1')* $Z_1 \subseteq Z_2 \Rightarrow Z_2^{\triangleleft} \subseteq Z_1^{\triangleleft}$
2) $Y \subseteq Y^{\triangleright\triangleleft}$	*2')* $Z \subseteq Z^{\triangleleft\triangleright}$
3) $Y^{\triangleright} = Y^{\triangleright\triangleleft\triangleright}$	*3')* $Z^{\triangleleft} = Y^{\triangleleft\triangleright\triangleleft}$

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4) $Y \subseteq Z^{\lhd} \Leftrightarrow Z \subseteq Y^{\rhd} \Leftrightarrow Y \times Z \subseteq R$

A *concept* is a pair (Y, Z) of objects and attributes that is evenly balanced in the sense that Z contains just those properties that pertain to all objects in Y, whereas Y is precisely the set of objects that have all the properties in Z:

Definition 2.6. *Let (W, X, R) be a context, let $Y \subseteq W$ be an extent and $Z \subseteq X$ an intent. Then (Y, Z) is a **concept** of (W, X, R) iff $Y^{\rhd} = Z$ and $Z^{\lhd} = Y$.*

The notational convention is to denote the set of all the concepts of a context (W, X, R) as $\mathfrak{B}(W, X, R)$. In the case where $W = R^1$ and $X = R^2$ this notation will henceforth be simplified to $\mathfrak{B}(R)$. The following lemmata are all standard and proofs are therefore omitted:

Lemma 2.7. *Each concept of a context (W, X, R) has the form $(Y^{\rhd\lhd}, Y^{\rhd})$ for some subset $Y \subseteq W$ and the form $(Z^{\lhd}, Z^{\lhd\rhd})$ for some subset $Z \subseteq X$. Conversely, all such pairs are concepts.*

Lemma 2.8. *If $(Y, Z) \in \mathfrak{B}(W, X, R)$ then $Y \times Z \subseteq W$.*

Lemma 2.9. $\{(\emptyset, \emptyset)\} = \mathfrak{B}(R)$ *iff* $R = \emptyset$

The next lemma is also useful, and since the present author is not aware of it being explicitly recorded anywhere, a verification is included.

Lemma 2.10 (Concept monotony). *If $Y \times Z \subseteq G$ then there is a $(Y', Z') \in \mathfrak{B}(G)$ such that $Y \subseteq Y'$ and $Z \subseteq Z'$.*

Proof. $Y \times Z \subseteq G$ entails $Z \subseteq Y^{\rhd}$ and $Y \subseteq Z^{\lhd}$ by theorem 2.5(4). By lemma 2.7 $((Z^{\lhd})^{\rhd\lhd}, (Z^{\lhd})^{\rhd}) \in \mathfrak{B}(G)$. Now, $Z^{\lhd} \subseteq (Z^{\lhd})^{\rhd\lhd}$ by 2.5(2) whence $Y \subseteq (Z^{\lhd})^{\rhd\lhd}$ by $Y \subseteq Z^{\lhd}$. Since $Z \subseteq (Z^{\lhd})^{\rhd} = Z^{\lhd\rhd}$ by 2.5(2'), the proof is complete.

□

Concepts can be ordered by extents or intents by putting $(Y_1, Z_1) \leq (Y_2, Z_2)$ either iff $Y_1 \subseteq Y_2$ or iff $Z_2 \subseteq Z_1$. The two alternatives give the same

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ordering since upper and lower adjoints of Galois connections are antitone (by theorem 2.5). The fundamental theorem of formal concept analysis says that the set of concept $\mathfrak{B}(W, X, R)$ of a context (W, X, R) forms a complete lattice under $\leq$ in which meet and join are given by,

$$\bigvee_{j \in J}(Y_j, Z_j) = \left((\bigcup_{j \in J} Y_j)^{\rhd\lhd}, \bigcap_{j \in J} Z_j \right)$$

$$\bigwedge_{j \in J}(Y_j, Z_j) = \left(\bigcap_{j \in J} Y_j, (\bigcup_{j \in J} Z_j)^{\lhd\rhd} \right)$$

Moreover, the relation R is encoded in, and can be read off from, the lattice $\mathfrak{B}(W, X, R)$. Put

$$\gamma(x) =_{df} (\{y\}^{\rhd\lhd}, \{y\}^{\rhd}) \text{ for any } y \in Y \text{ and}$$

$$\mu(z) =_{df} (\{z\}^{\lhd}, \{z\}^{\lhd\rhd}) \text{ for any } z \in Z$$

then $\gamma(y) \leq \mu(z)$ iff $(y, z) \in R$. Conversely, the extent and intent of any concept $\mathbf{c} \in \mathfrak{B}(W, X, R)$ can be read off using these same functions:

$$(Y_{\mathbf{c}}, Z_{\mathbf{c}}) := (\{x \in Y : \gamma(x) \leq \mathbf{c}\}, \{y \in Z : \mathbf{c} \leq \mu(y)\}) \tag{1}$$

Example 2.11. *Fig. 1 shows a cross-table for a binary relation R and the concept lattice it gives rise to. Every concept of form $(\{y\}^{\rhd}, \{y\}^{\rhd\lhd})$ for some object $w \in R^1$ has been labelled $\gamma(w)$, and every concept of form $(\{z\}^{\lhd}, \{z\}^{\lhd\rhd})$ for some attribute $z \in R^2$ has been labelled $\mu(z)$. The fundamental theorem of concept lattices says, essentially, that the extent and intent of any concept $\mathbf{c}$ in the lattice can be read off from the labels in $(\downarrow \mathbf{c}, \uparrow \mathbf{c})$ by letting every argument to μ be an intent and every argument to γ be an extent of the concept in question. For instance the concept labelled $\gamma(x_3)$ is $(\{x_2, x_3\}, \{x_2, y_1\})$. Note that it is not a requirement of concept lattices, although often assumed, that extents and intents be disjoint. This particular liberty is essential for the FCA approach to* basic reusable output *in section 4.*

3. Concept lattices and basic output.

The connection between concept lattices and input/output logic is based on giving concepts a logical interpretation: recall that (Y, Z) being a concept

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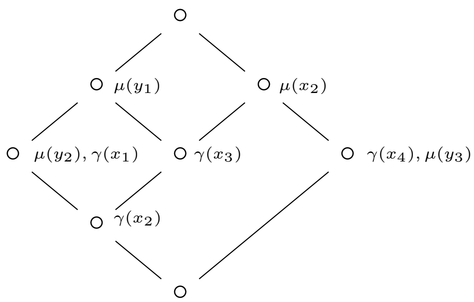

	x_2	y_2	y_3	y_4
x_1		×	×	
x_2	×	×	×	
x_3	×	×		
x_4	×			×

Figure 1: A context and its associated concept lattice

means that Y consists of the set of objects which is such that *each* object in Y can be ascribed *all* attributes in Z. When a concept (A, B) in question is determined by a binary relation G over *formulae*, it is natural to interpret this as saying that each formula in A constitutes a sufficient condition for every formula in B to obtain. By extension of this thought it is natural to understand extents *disjunctively* and intents *con*junctively.

Henceforth, this interpretation of concepts (restricted, of course, to concept lattices that are generated by relations over formulae) will be made explicit using a function ψ:

Definition 3.1. *$(A, B)^\psi =_{df} (\bigvee A, \bigwedge B)$, where $\bigvee \emptyset$ is defined as $\bot$ and $\bigwedge \emptyset$ as $\top$.*

Clearly, ψ as so defined is injective. For any set X of concepts $X^\psi =_{df} \{c^\psi : \psi \in X\}$.

As mentioned already, the pair $(\top, \top)$ will not be valid in the concept semantics for basic output that is to be developed in the present section. It therefore needs to be removed from the concept of derivability as well. From here on, derivability means:

Definition 3.2. *A **derivation** of a pair (a, b) from G, given a set X of rules, is understood to be a tree with (a, b) at the root, each non-leaf node related to its immediate parents by the inverse of a rule in X, and each leaf node an element of G.*

Bearing this adjustment in mind, the nomenclature from table 1 will be reused as is. For instance $\Vdash_2$ will continue to denote the input/output logic

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characterised by the set of rules $\{SI, AND, WO, OR\}$, only it is understood that derivability is according to definition 3.2.

Turning now to the first substantial theorem of the present paper, it says that the concept lattice induced by a relation G over formulae gives rise to a set of axioms that are equivalent to G modulo basic output:

Theorem 3.3. *If G is non-empty then $G \Vdash_2 (a, b)$ iff $\mathfrak{B}(G)^\psi \Vdash_2 (a, b)$.*

Proof. For the left-to-right direction, it suffices to show that $\mathfrak{B}(G)^\psi \Vdash_2 (c, d)$ for every $(c, d) \in G$, for if $G \Vdash_2 (a, b)$, then it follows by monotony for $\Vdash_2$ that $\mathfrak{B}(G)^\psi, G \Vdash_2 (a, b)$ whence $\mathfrak{B}(G)^\psi \Vdash_2 (a, b)$ by cumulative transitivity for $\Vdash_2$. So suppose $(c, d) \in G$. We have $(\{c\}^{\rhd\lhd}, \{c\}^{\rhd}) \in \mathfrak{B}(G)$ by lemma 2.7, and $c \in \{c\}^{\rhd\lhd}$ by theorem 2.5(2). Therefore, by reflexivity and thinning for $\vdash$ it follows that $c \vdash \{c\}^{\rhd\lhd}$, whence $c \vdash \bigvee\{c\}^{\rhd\lhd}$ by $\vee$-classicality. As regards $\{c\}^{\rhd}$ we have $d \in \{c\}^{\rhd}$ since $(c, d) \in G$. Since $(\bigvee\{c\}^{\rhd\lhd}, \bigwedge\{c\}^{\rhd}) \in \mathfrak{B}(G)^\psi$, by the definition of ψ, successive applications of WO and SI now yields $\mathfrak{B}(G)^\psi \Vdash_2 (c, d)$ as desired. For the converse inclusion, it suffices, by reasoning similar to that above, to show that $(c, d) \in \mathfrak{B}(G)^\psi$ implies $G \Vdash_2 (c, d)$. Suppose $(\bigvee C, \bigwedge D) \in \mathfrak{B}(G)^\psi$ and thus by the injectiveness of ψ that $(C, D) \in \mathfrak{B}(G)$. For the limiting case that $C = \emptyset$ we have $\bigvee C = \perp$. Since G is non-empty by assumption it follows by lemma 2.9 that $D \neq \emptyset$. Thus there are pairs $(e_k, d_k) \in G$ for $k \in K$ such that $\bigwedge_{k \in K} d_k \dashv\vdash D$. By SI we have $G \Vdash_2 (\perp, d_k)$ for $k \in K$, whence $G \Vdash_2 (\perp, \bigwedge_{k \in K} d_k)$ by AND. Since $C = \emptyset$ we have $\bigvee C = \perp$ so $G \Vdash_2 (\bigvee C, \bigwedge D)$ by SI as desired. For the limiting case that $D = \emptyset$ we have $\bigwedge D = \top$. Since G is non-empty by assumption it follows by lemma 2.9 that $C \neq \emptyset$. Thus there are pairs $(e_k, d) \in G$ for $k \in K$ such that $\bigvee_{k \in K} e_k \dashv\vdash C$. By WO we have $G \Vdash_2 (e_k, \top)$ for $k \in K$, whence $G \Vdash_2 (\bigvee_{k \in K}, e_k)$ by OR. Since $D = \emptyset$ we have $\bigwedge D = \top$ so $G \Vdash_2 (\bigvee C, \bigwedge D)$ by WO as desired. For the principal case that $C \neq \emptyset \neq D$ suppose that $C = \{c_1, \ldots, c_n\}$ and $D = \{d_1, \ldots, d_k\}$. Since $D = C^{\rhd}$ we have $D \subseteq G(c_i)$ for every $1 \leq i \leq n$, whence $(c_i, d_j) \in G$ for every $1 \leq j \leq k$. It follows by AND that $(c_i, \bigwedge_{j=1}^{k} d_k)$ is $\Vdash_2$-derivable from G. Therefore $(\bigvee_{i=1}^{n} c_i, \bigwedge D)$ is derivable from G by OR whence $(\bigvee C, \bigwedge D)$ is derivable from G by SI.

□

Note that the correspondence breaks down in the limiting case that G is empty, for then $(\emptyset, \emptyset) \in \mathfrak{B}(G)$, by lemma 2.9, whence $(\perp, \top) \in \mathfrak{B}(G)^\psi$

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by definition 3.1. Yet, $\emptyset \not\Vdash_2 (\bot, \top)$. The qualification to non-empty G is therefore essential, and recurs throughout.

Whereas theorem 3.3 gives $\Vdash_2$-equivalent lattice representation for a set of input/output axioms G, it is not yet a semantics since what the lattice implies is specified with reference to $\Vdash_2$. The question is whether it is possible to push the envelope and use the concept lattice itself as a semantic structure against which to evaluate pairs of propositions. As it turns out, this question has an affirmative answer.

Definition 3.4. *Let $\mathbb{E}_G(a)$ denote the set of concepts in $\mathfrak{B}(G)$ whose extents are each entailed by a. That is,*

$$\mathbb{E}_G(a) =_{df} \{(A, B) \in \mathfrak{B}(G) \mid a \vdash A\}$$

We have:

Theorem 3.5. *Assume that G is non-empty and put $\mathbb{E}_G(a) := \{(A_i, B_i)\}_{i\in I}$. Define an operation $\mathbb{O}_2$ as follows:*

$$(a, b) \in \mathbb{O}_2(G) \textit{ iff } \{B_i\}_{i\in I} \vdash b$$

Then $(a, b) \in \mathbb{O}_2(G)$ iff $G \Vdash_2 (a, b)$.

Proof. For the right-to-left direction, we prove by induction on the derivation of (a, b) from G, using only the stipulated proof rules, that $(a, b) \in \mathbb{O}_2(G)$. In the base case $(a, b) \in G$. Since $a \in a^{\rhd\lhd}$ it follows that $(a^{\rhd\lhd}, a^{\rhd}) = (A_k, B_k)$ for some $k \in I$. Therefore since $b \in a^{\rhd}$ we have $\{B_i\}_{i\in I} \vdash b$ as desired. For input strengthening suppose (a, b) is derived from (a', b) by SI. Then by the induction hypothesis there is a family of concepts $\{(A_i, B_i)\}_{i\in I}$ with $a' \vdash A_i$ for all $i \in I$ and $\{B_i\}_{i\in I} \vdash b$. It thus suffices to show that $a \vdash A_i$ for all $i \in I$, which follows from $a \vdash a'$ by cut. For output weakening suppose (a, b) is derived from (a, b') by WO. Then by the induction hypothesis there is a family of concepts $\{(A_i, B_i)\}_{i\in I}$ with $a \vdash A_i$ for all $i \in I$ and $\{B_i\}_{i\in I} \vdash b'$. Since $b' \vdash b$ it thus follows by cut that $\{B_i\}_{i\in I} \vdash b$ and we are done. For AND suppose $(a, b_1 \wedge b_2)$ is derived from (a, b_1) and (a, b_2). By the induction hypothesis there are families of concepts $\{(A_i, B_i)\}_{i\in I}$ and $\{(A_j, B_j)\}_{j\in J}$ such that $a \vdash A_k$ for every $k \in I \cup J$ with $\{B_i\}_{i\in I} \vdash b_1$ and $\{B_j\}_{j\in J} \vdash b_2$. It follows by $\wedge$-classicality, thinning and cut that $\{B_i\}_{i\in I} \cup \{B_j\}_{j\in J} \vdash b_1 \wedge b_2$ whence the family of concepts $\{(A_k, B_k)\}_{k\in I\cup J}$ has the required properties. For OR

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suppose $(a_1 \vee a_2, b)$ is derived from (a_1, b) and (a_2, b). By the induction hypothesis there are families of concepts $\{(A_i, B_i)\}_{i \in I}$ and $\{(A_j, B_j)\}_{j \in J}$ such that $a \vdash A_k$ for every $k \in I \cup J$ with $\{B_i\}_{i \in I} \vdash b$ and $\{B_j\}_{j \in J} \vdash b$. Since by thinning $\{B_k\}_{k \in I \cup J} \vdash b$ it follows that the family of concepts $\{(A_k, B_k)\}_{k \in I \cup J}$ has the required properties.

For the converse direction, suppose $(a, b) \in \mathbb{O}_2(G)$. Then $a \vdash A_i$ for each $i \in I$ and $\{B_i\}_{i \in I} \vdash b$. It suffices to show for any $j \in I$ that $(a, \bigwedge B_j)$ is derivable from G using only the stipulated proof rules, because then $(a, \bigwedge_{i \in I}(\bigwedge B_i))$ is derivable by AND, from which it follows in turn that (a, b) is derivable by $\wedge$-classicality and WO. Thus, consider any (A_j, B_j) such that $j \in I$. There are four cases to consider:

1. $A_j = \emptyset$, $B_j \neq \emptyset$: Then since $a \vdash A_j$ we have $a \vdash \emptyset$ and so $a \vdash \bot$. Since $B_j \neq \emptyset$ there are pairs $(e_1, b_1), \ldots, (e_n, b_n) \in G$ such that $\{b_i\}_{1 \leq i \leq n} = B_j$. We thus have the following derivation

$$\dfrac{\dfrac{\dfrac{(e_1, b_1), \ldots, (e_n, b_n)}{(\bot, b_1), \ldots, (\bot, b_n)}\,SI}{(\bot, \bigwedge_{i=1}^n b_i)}\,AND}{(a, \bigwedge_{i=1}^n b_i)}\,SI$$

   It follows that $(a, \bigwedge B_j)$ is derivable from G as desired.

2. $A_j \neq \emptyset$, $B_j = \emptyset$: We wish to show that $(a, \bigwedge B_j)$ is derivable from G. Since $B_j = \emptyset$ we have $\bigwedge B_j = \top$, so it suffices to show the derivability of $(a, \top)$. Since $A_j \neq \emptyset$ there are pairs $(a_1, e_1), \ldots, (a_n, e_n) \in G$ such that $\{a_i\}_{1 \leq i \leq n} = A_j$. We therefore have the following derivation:

$$\dfrac{\dfrac{\dfrac{(a_1, e_1), \ldots, (a_n, e_n)}{(a_1, \top), \ldots, (a_n, \top)}\,WO}{(\bigvee_{i=1}^n a_i, \top)}\,OR}{(a, \top)}\,SI \quad \text{Since } a \vdash A_j \text{ iff } a \vdash \textstyle\bigvee_{i=1}^n a_i \text{ by } \vee\text{-classicality}$$

3. $A_j \neq \emptyset$, $B_j \neq \emptyset$: Since (A_j, B_j) is a concept of $\mathfrak{B}(G)$ it follows by lemma 2.8 that $A_j \times B_j \subseteq G$. Put $A_j := \{a_i\}_{1 \leq i \leq n}$, then in particular we have $\{a_i\} \times B_j \subseteq G$ for each i. Therefore, $(a_i, \bigwedge B_j)$ is derivable from G by repeated applications of AND. Thus, taking the $(a_i, \bigwedge B_j)$ as leaves we have the following derivation from G:

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$$OR \frac{(a_1, \bigwedge B_j), \ldots, (a_n, \bigwedge B_j)}{SI \dfrac{(\bigvee_{i=1}^{n} a_i, \bigwedge B_j)}{(a, \bigwedge B_j)} \text{ Since } a \vdash A_j \text{ iff } a \vdash \bigvee_{i=1}^{n} a_i \text{ by } \vee\text{-classicality}}$$

4. $A_j = \emptyset = B_j$: This case is ruled out by lemma 2.9 and the assumption that G is non-empty.

$\square$

Theorem 3.5 can be taken to show that not only can a set of axioms for basic input/output logic be visualised as a concept lattice, the converse is true as well: basic output has a reasonable claim to be called the logic of concept lattices.

Example 3.6. *A national labour and welfare service provides social benefits that contribute to the financial security of citizens and other right holders. Being entitled to a social benefit usually depends on certain required pieces of information that are associated by law with the tasks in question. Assume the following lists of tasks and information requirements:*

Tasks:	***Information requirements:***
a = *declare incapacity to work*	g = *verify sickness*
b = *grant sickness pay*	h = *determine line of work*
c = *fund rehabilitation*	i = *determine present education*
d = *fund education*	j = *obtain prognosis*

Let the relationship between tasks and information requirements be given by the cross-table in fig 2. Then e.g. the element labelled $\gamma(a)$ in the associated concept lattice tells us that declaration of incapacity to work requires verification of sickness and *line of work.*

As regards the operation $\mathbb{O}_2$, consider the query "is it so that declaring incapacity to work and *funding rehabilitation requires determining present line of work?". This amounts to asking whether the pair $(a \wedge c, h)$ is semantically valid according to the definition of $\mathbb{O}_2$. The query can be answered by considering the concepts labelled $\gamma(c)$ and $\gamma(a)$ respectively, these are $\{\mathbf{c}_c := (\{c\}, \{j, h\}), \mathbf{c}_a := (\{a\}, \{g, h\})\} = \mathbb{E}_G(a \wedge c)$. Since $\{g, h\}, \{j, h\} \vdash h$, the answer to the query is affirmative.*

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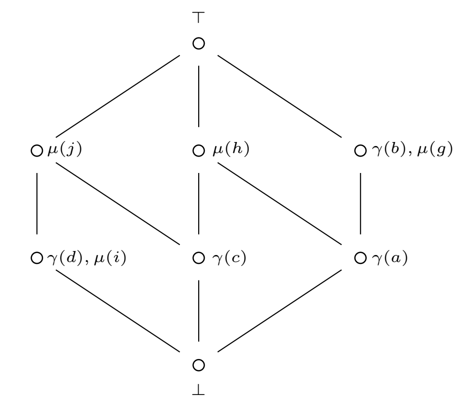

	g	h	i	j
a	×	×		
b	×			
c		×		×
d			×	×

Figure 2: A concept lattice for (some) social benefits.

4. Concept lattices and basic reusable output.

The previous section showed that any set of axioms for a system of basic output can A) faithfully be represented as a concept lattice (theorem 3.3), and B) that this concept lattice constitutes a semantic structure against which any pair of formulae can be evaluated wrt. membership in the logic in question (theorem 3.5). Taken together, these results offer a way of 'mining' a system of basic output using a blend of lattice theoretic techniques and propositional logic.

It is an interesting question whether a similar result can be established for the stronger, and for certain applications more interesting (cf. section 5) system of basic *reusable* output. The two parts of this question will continue to be referenced as A) and B) in the following, bearing in mind that it is now derivability according to $\Vdash_4$ which is at issue.

Care needs to be taken when interpreting A), since it has in one sense already been answered affirmatively: theorem 3.3 showed that $G \Vdash_2 (a, b)$ iff $\mathfrak{B}(G)^\psi \Vdash_2 (a, b)$. Since cumulative transitivity is a Horn rule, it follows that $G \Vdash_4 (a, b)$ iff $\mathfrak{B}(G)^\psi \Vdash_4 (a, b)$. Thus, representation of any $\Vdash_4$-system by *some* lattice is immediate. It should be obvious, though, that no ground is gained by this observation if the aim is to analyse the *reuse* feature of reusable basic output using lattice theoretic techniques, for the lattice representation $\mathfrak{B}(G)^\psi$ is the same as for basic output, whereas the logic of $\Vdash_4$ is stronger. In other words, none of the information that is specific to $\Vdash_4$ is explicitly present in $\mathfrak{B}(G)^\psi$.

Better, then, to look for a lattice that, so to speak, wears its $\Vdash_4$-inferences

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(at least some of them) on its sleeves. This is the strategy that will be pursued in the present section. What is sought is thus a principled way to extend G into a larger G^+ such that $G \Vdash_4 (a, b)$ iff $\mathfrak{B}(G^+)^\psi \Vdash_4 (a, b)$. To be sure, for any such G^+ it follows that $\mathfrak{B}(G^+)^\psi \Vdash_4 (a, b)$ iff $\mathfrak{B}(G)^\psi \Vdash_4 (a, b)$ so $\mathfrak{B}(G^+)$ will have the same inferential potential as $\mathfrak{B}(G)$ itself modulo $\Vdash_4$. However, $\mathfrak{B}(G^+)$ and $\mathfrak{B}(G)$ *is not the same lattice*, and this is what matters for present purposes.

In what follows, G^+ will be built from G by a process called saturation—not to be confused with any of the other things called saturation out there—which can be thought of as the counterpart to the notion of transitive closure within the FCA framework.

Definition 4.1. *For any $G \subseteq L^2$ the* ***saturation*** *G^+ of G is the least relation that contains G and is closed under the rule*

$$\mathbf{S}: \qquad \textit{If } (A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G^+) \textit{ and } B_1 \vdash A_2 \textit{ then } A_1 \times B_2 \subseteq G^+$$

The following equivalent inductive definition tends to be more convenient in proofs:

Definition 4.2. $G_+ := \bigcup_{i=0}^{\omega} G_i$, *where*

1. $G_0 = G$
2. $G_{n+1} = G_n \cup (\bigcup\{A_1 \times B_2 \mid (A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G_n) \textit{ and } B_1 \vdash A_2\})$

Lemma 4.3. $G^+ = G_+$

Proof. To show that $G^+ \subseteq G_+$ it suffices to show that G_+ includes G and that it is is closed under the rule $\mathbf{S}$. The former follows immediately from case 1 of definition 4.2. For the latter, put $G_+ := \bigcup\{G_i\}_{i<\omega}$ and suppose $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G_+)$ and $B_1 \vdash A_2$. We need to show that $A_1 \times B_2 \subseteq G_+$. From the first assumption we have $(A_1 \times B_1) \cup (A_2 \times B_2) \subseteq G_+$ by lemma 2.8. Since $\{G_i\}_{i<\omega}$ is a directed set, there is a finite k such that

a) $(A_1 \times B_1) \subseteq G_k$

b) $(A_2 \times B_2) \subseteq G_k$

We can thus infer the following about G_k:

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$\rightarrow \mathbf{1}$: $(A_1^{\rhd\lhd}, A_1^{\rhd}), (A_2^{\rhd\lhd}, A_2^{\rhd}) \in \mathfrak{B}(G_k)$ by lemma 2.7, since $A_1, A_2 \subseteq (G_k)^1$ by a) and b).

$\rightarrow \mathbf{2}$: $B_2 \subseteq A_2^{\rhd}$ from b).

$\rightarrow \mathbf{3}$:$B_1 \subseteq A_1^{\rhd}$ from a)

In light of the fact that $A_1 \subseteq A_1^{\rhd\lhd}$ (2.5), item $\rightarrow \mathbf{2}$ entails $A_1 \times B_2 \subseteq A_1^{\rhd\lhd} \times A_2^{\rhd}$. Now, $B_1 \vdash A_2$ by assumption. Therefore item $\rightarrow \mathbf{3}$ taken together with $A_2 \subseteq A_2^{\rhd\lhd}$ (theorem 2.5) entails $A_1^{\rhd} \vdash A_2^{\rhd\lhd}$ by thinning. But then $A_1^{\rhd\lhd} \times A_2^{\rhd} \subseteq G_{k+1}$, by $\rightarrow \mathbf{1}$ together with case 2 of definition 4.2, whence $A_1 \times B_2 \subseteq G^+$ by the transitivity of $\subseteq$ as desired.

For the converse direction it suffices to prove by induction on $n < \omega$ that $G_n \subseteq G^+$. This is immediate for the base case where $n = 0$. For the induction step, assume as the induction hypothesis that $G_{n-1} \subseteq G^+$. We need to show that $G_n \subseteq G^+$. Suppose not. Then by case 2 of definition 4.2 there are $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G_{n-1})$ with $B_1 \vdash A_2$ such that $A_1 \times B_2 \nsubseteq G^+$. By the induction hypothesis together with lemma 2.8 we have

i) $(A_1 \times B_1) \subseteq G^+$ and

ii) $(A_2 \times B_2) \subseteq G^+$.

We can thus infer the following about G^+:

$\leftarrow \mathbf{1}$: $(A_1^{\rhd\lhd}, A_1^{\rhd}), (A_2^{\rhd\lhd}, A_2^{\rhd}) \in \mathfrak{B}(G^+)$ by lemma 2.7, since $A_1, A_2 \subseteq (G^+)^1$ by i) and ii).

$\leftarrow \mathbf{2}$: $B_2 \subseteq A_2^{\rhd}$ from ii).

$\leftarrow \mathbf{3}$: $B_1 \subseteq A_1^{\rhd}$ from i)

In light of the fact that $A_1 \subseteq A_1^{\rhd\lhd}$, by theorem 2.5, item $\leftarrow \mathbf{2}$ above entails $A_1 \times B_2 \subseteq A_1^{\rhd\lhd} \times A_2^{\rhd}$. Now, $B_1 \vdash A_2$ by assumption. Therefore item $\leftarrow \mathbf{3}$ taken together with $A_2 \subseteq A_2^{\rhd\lhd}$ (theorem 2.5) entails $A_1^{\rhd} \vdash A_2^{\rhd\lhd}$ by thinning. But then $(A_1^{\rhd\lhd}, A_2^{\rhd}) \in G^+$ by $\rightarrow \mathbf{3}$ and the rule **S**. Therefore $(A_1^{\rhd\lhd} \times A_2^{\rhd}) \subseteq G^+$ whence $A_1 \times B_2 \subseteq G^+$ by the transitivity of $\subseteq$. This contradicts the assumption that $A_1 \times B_2 \nsubseteq G^+$.

□

Note that G^+ as so defined exists for any G:

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Lemma 4.4. *If $\mathbb{G}$ is a set of saturated super-sets of G then $\bigcap \mathbb{G}$ is a saturated super-set of G.*

Proof. Suppose $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(\bigcap \mathbb{G})$ such that $B_1 \vdash A_2$. Let G be any member of $\mathbb{G}$. Then $\bigcap \mathbb{G} \subseteq G$ from which it follows by lemma 2.10 that there are $(C_1, D_1), (C_2, D_2) \in \mathfrak{B}(G)$ such that $A_i \subseteq C_i$ and $B_i \subseteq D_i$ for $i \in \{1, 2\}$. By thinning $B_1 \vdash A_2$ implies $D_1 \vdash C_2$, whence $C_1 \times D_2 \subseteq G$ by the assumption that G is saturated. Since $A_1 \times B_2 \subseteq C_1 \times D_2$ it follows that $A_1 \times B_2 \subseteq G$. Since G was chosen arbitrarily it follows that $A_1 \times B_2 \subseteq \bigcap \mathbb{G}$.

□

Corollary 4.5. *Every binary relation G has a least saturated super-relation G^+.*

The existence of least fixpoints for the saturation operation provides an answer to A) that is informative wrt. $\Vdash_4$:

Theorem 4.6. *If G is non-empty then $G \Vdash_4 (a, b)$ iff $\mathfrak{B}(G^+)^\psi \Vdash_4 (a, b)$.*

Proof. The left-to-right inclusion is exactly like that for theorem 3.3 and thus requires the non-emptiness of G.

For the converse inclusion, it suffices by monotony, cumulative transitivity for $\Vdash_4$ together with lemma 4.3 to show that $(a, b) \in \mathfrak{B}(G_+)^\psi$ implies $G \Vdash_4 (a, b)$. So, put $G_+ := \bigcup \{G_i\}_{i<\omega}$. Since G is finite it follows from definition 4.2 that $G_+ = G_n$ for some finite n. We prove by induction n that the desired implication follows from $G_+ = G_n$ on any value of n.

The proof of the base case reduces to the right-to-left inclusion of theorem 3.3. For the induction step suppose the implication holds for $n - 1$. Let $(a, b) := (\bigvee A, \bigwedge B)$ and suppose $(a, b) \in \mathfrak{B}(G_n)^\psi$. We need to show that $G \Vdash_4 (a, b)$. Since ψ is injective, we have $(A, B) \in \mathfrak{B}(G_n)$. If $(A, B) \in \mathfrak{B}(G_{n-1})$ then $(\bigvee A, \bigwedge B) \in \mathfrak{B}(G_{n-1})^\psi$ and therefore $G \Vdash_4 (a, b)$ by the induction hypothesis. Suppose therefore that the opposite is the case, i.e. that $(A, B) \notin \mathfrak{B}(G_{n-1})$. There are now two cases to consider:

1. $A \times B \subseteq G_{n-1}$: Then by lemma 2.10 there is a concept $(A', B') \in \mathfrak{B}(G_{n-1})$ with $A \subseteq A'$ and $B \subseteq B'$. We have $(\bigvee A', \bigwedge B') \in \mathfrak{B}(G_{n-1})^\psi$ and so $G \Vdash_4 (\bigvee A', \bigwedge B')$ by the induction hypothesis. Since $a = \bigvee A \vdash \bigvee A'$ and $\bigwedge B' \vdash \bigwedge B = b$ we may apply SI and WO to conclude that $G \Vdash_4 (a, b)$ as desired.

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2. $A \times B \nsubseteq G_{n-1}$: It suffices to show that for every $c \in A$ and every $d \in B$ we have $G \Vdash_4 (c, d)$ since then $G \Vdash_4 (c, \bigwedge B)$, follows by AND and $G \Vdash_4 (\bigvee A, \bigwedge B)$ by OR, whence $G \Vdash_4 (a, b)$ as desired. So let c be any element in A and d any element in B. By the supposition of the case we have $A \times B \nsubseteq G_{n-1}$ whilst $(A, B) \in \mathfrak{B}(G_n)$. If $(c, d) \in G_{n-1}$ then by lemma 2.7 $(\{c\}^{\rhd\lhd}, \{c\}^{\rhd}) \in \mathfrak{B}(G_{n-1})$ whence $(\bigvee\{c\}^{\rhd\lhd}, \bigwedge\{c\}^{\rhd}) \in \mathfrak{B}(G_{n-1})^{\psi}$ by definition 3.1. Therefore $G \Vdash_4 (c, d)$ by the induction hypothesis together with WO and SI. If $(c, d) \notin G_{n-1}$ then since $(c, d) \in A \times B \subseteq G_n$, by lemma 2.10, it follows by the definition of G_n that there are $(C, E_1), (E_2, D) \in \mathfrak{B}(G_{n-1})$ with $E_1 \vdash E_2$, $c \in C$ and $d \in D$. Since $(\bigvee C, \bigwedge E_1), (\bigvee E_2, \bigwedge D) \in \mathfrak{B}(G_{n-1})^{\psi}$, by the definition of ψ, it follows by the induction hypothesis that $(\bigvee C, \bigwedge E_1), (\bigvee E_2, \bigwedge D)$ are $\Vdash_4$-derivable from G . We therefore have the following derivation:

$$
\dfrac{\dfrac{\dfrac{}{(\bigvee C, \bigwedge E_1)}\,\text{hypothesis} \qquad \dfrac{\dfrac{\dfrac{}{(\bigvee E_2, \bigwedge D)}\,\text{hypothesis}}{(\bigwedge E_1, \bigwedge D)}\,SI \text{ since } E_1 \vdash E_2}{(\bigvee C \wedge \bigwedge E_1, \bigwedge D)}\,SI}{\dfrac{(\bigvee C, \bigwedge D)}{\dfrac{(\bigvee C, d)}{(c, d)}\,SI \text{ since } c \in C}\,WO \text{ since } d \in D}\,CT
$$

This completes the proof.

□

Turning now to B) the question is whether this concept lattice, where for any given G the anaphor 'this' is bound by the grammatical antecedent $\mathfrak{B}(G^+)$ gives a semantics (in the broad sense of the term) for the logic $C_4(G)$. That is, the question is whether there is an evaluation rule similar to $\mathbb{O}_2$ of previous section that given any pair (a, b) acts as the characteristic function for $C_4(G)$.

The first thing to note in this connection is that the $\mathbb{O}_2$ *applied to* G^+ is itself *sound* wrt. $\Vdash_4$-derivability from G. The proof of this property requires a lemma:

Lemma 4.7. *If* $G^+ \Vdash_4 (a, b)$ *then* $G \Vdash_4 (a, b)$.

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Proof. By idempotence and monotony it suffices to show that $G^+ \subseteq C_4(G)$. Put $G^+ := \bigcup\{G_i\}_{i<\omega}$, which by theorem 4.3 is legitimate. The proof proceeds by induction on n. The base case where $n = 0$ is immediate. For the induction step, suppose the property holds for $n-1$ and suppose $(a,b) \in G_n$. If $(a,b) \in G$ then there is nothing to prove, so suppose the opposite. Then by case 2 of definition 4.2 there are $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G_{n-1})$ with $B_1 \vdash A_2$ such that $a \in A_1$ and $b \in B_2$. By the induction hypothesis we have $A_i \times B_i \subseteq C_4(G)$ for $i \in \{1, 2\}$. By repeated applications of AND and OR it follows that $(\bigvee A_i, \bigwedge B_i) \subseteq C_n(G)$. Since $B_1 \vdash A_2$ it follows that $\bigwedge B_1 \vdash \bigvee A_2$. Therefore $(\bigvee A_1, \bigwedge B_2) \in C_4(G)$ by CT. Now, since $a \in A_1$ it follows that $a \vdash \bigvee A_1$ and since $b \in B_2$ it follows that $\bigwedge B_2 \vdash b$. Hence $(a,b) \in C_4(G)$ by SI and WO as desired.

□

As for the property itself:

Theorem 4.8. *If* $(a,b) \in \mathbb{O}_2(G^+)$ *then* $G \Vdash_4 (a,b)$

Proof. By theorem 3.5, $(a,b) \in \mathbb{O}_2(G^+)$ implies $G^+ \Vdash_2 (a,b)$, moreover $G^+ \Vdash_2 (a,b)$ implies $G^+ \Vdash_4 (a,b)$, since CT is a Horn condition, and by lemma 4.7 we have that $G^+ \Vdash_4 (a,b)$ implies $G \Vdash_4 (a,b)$. Chaining implications now yields the desired result.

□

Theorem 4.8 says that the lattice $\mathfrak{B}(G^+)$ may safely be 'mined' for entailments by selecting sets of concepts $\mathbb{E}_{G^+}$ as specified in definition 3.5—no unlicensed inferences can come from this.

It would have been nice if one could show the converse too, because one would then have a reduction of $\Vdash_4$ to $\Vdash_2$ in *finite* concept lattices. However, the converse of theorem 4.8—the completeness direction—does not hold. To see this, put $G := \{(a, b_1), (c, b_2), (b_1 \wedge b_2, d)\}$. Then $\mathfrak{B}(G)$ is the lattice in fig. 3.

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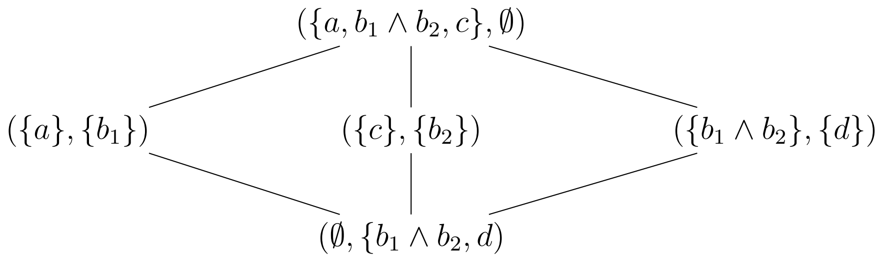

Figure 3: The lattice induced by G

Notice that G is already saturated—no intent entails another extent (recall that $\emptyset$ on the right of $\vdash$ means $\bot$). Notice also that $\mathbb{E}_G(a \wedge c) = \{(\{a, b_1 \wedge b_2, c\}, \emptyset), (\{a\}, \{b_1\}), (\{c\}, \{b_2\})\}$ but $\{b_1\}, \{b_2\} \nvdash d$. In other words $(a \wedge c, d) \notin \mathbb{O}_2(G)$. Yet, $G \Vdash_4 (a \wedge c, d)$ as the following derivation shows:

$$
CT\,\dfrac{AND\,\dfrac{SI\,\dfrac{(a, b_1)}{(a \wedge c, b_1)} \qquad SI\,\dfrac{(c, b_2)}{(a \wedge c, b_2)}}{(a \wedge c, b_1 \wedge b_2)} \qquad SI\,\dfrac{(b_1 \wedge b_2, d)}{(a \wedge c \wedge b_1 \wedge b_2, d)}}{(a \wedge c, d)}
$$

Since G is in fact saturated, it follows that $\mathbb{O}_2$ is *not* a $\Vdash_4$-complete evaluation rule wrt. saturated relations in general. In other words $G \Vdash_4 (a, b)$ does not entail $(a, b) \in \mathbb{O}_2(G^+)$, and so does not *a fortiori* entail $\mathfrak{B}(G^+)^\psi \Vdash_2 (a, b)$. The saturation procedure is just not sufficiently exhaustive to relieve the input/output processor of the burden of reusing outputs as inputs.

There are two general ways to respond to this situation: one is to come up with a saturation process—'saturation' now being understood in the intuitive sense of a way of amplifying the relation G with pairs that are derivable from G by CT—that is sufficiently complete to make the $\mathbb{O}_2$ rule to work for basic reusable output as well. The second option is of course to define a separate, stronger evaluation rule $\mathbb{O}_4$.

As regards the first option, one might be tempted to consider $C_4(G)$ itself a candidate saturated super relation of G. But this would be a sterile move, for even if it were the case that $G \Vdash_4 (a, b)$ iff $\mathfrak{B}(C_4(G))^\psi \Vdash_2 (a, b)$, which seems likely, this could hardly have been called a semantics since $\Vdash_4$ occurs on both sides of the biconditional. Moreover, if the underlying logical language L is not logically finite then $\mathfrak{B}(C_4(G))$ is an infinite concept lattice, so the reduction alluded to above would have lost its luster.

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As regards the second option, it is not obvious that $\mathbb{O}_2$ can be strengthened to take up the slack in a way that is *non-trivial*. That is, although something along the lines of $(a,b) \in \mathbb{O}_4(G)$ iff there is a $c \in L$ such that $(a,c),(a \wedge c, b) \in \mathbb{O}_2(G)$ would work, this too seems a rather unexciting prospect hardly deserving of being called a semantics. Yet, something very much like it is probably required.

The attempt to negotiate an answer to B) in the context of basic reusable output has reached an impasse that at this point is better left for future research. Taking stock, the situation now is this: Every binary relation G over L has a $\Vdash_4$-equivalent representation as a lattice of concepts (theorem 4.6). Moreover, the information contained in $\mathfrak{B}(G)$ can be processed with the operator $\mathbb{O}_2$ without risk of logical distortion (theorem 4.8). Although, this procedure will not in general produce all the information that is encoded in $\mathfrak{B}(G^+)$ relative to $\Vdash_4$, the next section argues that it is enough to secure interesting applications.

5. Extended example

A legal corpus is an eminent example of a relation between conditions and consequences that it is not natural to subsume under the inference paradigm, and/or to treat as a set of conditionals. As Hans Kelsen argued, norms are better seen as stipulations, laid down by some authority for some purpose, that are logically arbitrary: "Norms posited by human acts of will are arbitrary in the genuine signification of the word: that is, they can decree any behaviour whatsoever to be obligatory" [8, p. 4].

The purpose of the present section is to illustrate the concept representation of input/output logic developed in the preceding sections by analysing a hypothetical but not unrealistic example of statutory law. The analysis is centered on the role of the principle of transitivity as a rule that serves to link separately maintained areas of law into a cohesive whole. The transitivity of normative implicature, one might say, is a principle that allows law to be modular.

By an area of law—fuzziness admitted—will here be meant some set of legal norms (written or otherwise) that approaches a distinct social institution. That is, it is practised by specialists, it has a more or less clearly demarcated legal subject matter, it evolves in relative independence from other areas of law, and so on and so forth. By way of example, property law defines

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the conditions under which rights and obligations pertaining to the control of physical and intangible objects are transferred between legal persona, whereas criminal law sets out the punishment to be imposed on behaviour that threatens, harms or endangers the safety and welfare of individuals or the society as a whole. Administrative law governs eligibility to hold office as well as the activities of administrative agencies of government, whereas banking law subject banks to requirements that create transparency between banking institutions and the individuals and corporations with whom they conduct business, etc.

The idea that is proposed in the present section is that it is by recognition of the transitivity of normative implicature that these different areas of law make up a cohesive whole. That is, when the law as such recognizes the general principle of chaining norms it becomes possible for one area of law to reuse legal consequences from another area of law as applicability conditions or legal provisos for its own regulatives.

Penal law occupies a central position in this edifice insofar as the sanctions imposed by penal law is an important determinant for the rights and obligations that can be ascribed to a legal person subject to another area of law. It is thus natural to take penal law as the starting point for the analysis to follow.

Note first that penal law usually distinguishes between different categories of unwanted behaviour. For instance, felonies vs. misdemeanours and criminal offences vs. violations of civic duties. The stipulated punishments typically vary with the seriousness and type of the transgression. Common classifications of sanctions comprise a) various forms of incarceration (imprisonment, mandatory reformatory psychiatry, b) loss or suspension of civil rights, c) reprimand/official reproof, d) removal from office/termination of employment, and e) fines.

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Suppose that the penal code in question correlates offenses as in table 3. The offenses themselves are listed in table 2:

Offenses:	
mi	= allowing identity documents to be misused
gl	= grand larceny
ll	= lesser larceny
r	= robbery/obtaining of property by threat of force
ltr	= lesser tax evasion
aa	= aggravated assault
le	= lesser embezzlement/dishonestly witholding assets
asa	= aggravated sexual assault
rca	= repeated child abuse
nda	= neglect of duty to assist a person who has fallen into peril
ht	= high treason/treason against the state
gm	= gross misconduct in public office

Sanctions:	
N**m**	= imprisonment of a maximum lenght of N months
N**y**	= imprisonment of a maximum lenght of N years
f	= fine
or	= reprimande, official reproof
rp	= incarceration and mandatory reformatory psychiatry
ii	= indeterminate imprisonment
fl	= financial liability towards the offended
ls	= loss of suffrage

Table 2: List of offenses

It should be emphasized at this point that, as stated, the offenses and punishments in table 2 are not sentences, but rather predicates and open sentences that can in general be instantiated by different agents. The correlation of such predicates is not in general free of problems since it may involve e.g. considerations of arity (cf. [9]).[1] Yet, going into detail about this would cloud the overall picture of the role of the principle transitivity in a modular corpus of norms, and it is uncertain whether the gains would be worth the investment. For now, therefore, the reader is simply invited to mentally prepend 'x is guilty of ...' and 'x is sentenced to ...', or something like it, wherever appropriate.

[1]We are grateful to one of the anonymous reviewers for raising this point.

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	6m	**10y**	**21y**	**f**	**fl**	**ls**	**rp**	**ii**	**or**
mi	×			×					
gl					×				
ll				×	×				
r					×				
lte	×					×			
aa					×				
le				×	×				
asa		×			×				
rca					×		×	×	
nda	×								
ht			×			×			
gm									×

Table 3: A hypothetical penal code.

Now, given the correlations in table 3, theorem 3.3 tells us that the lattice in fig. 4 gives an $\Vdash_2$-equivalent representation of the penal code in question:

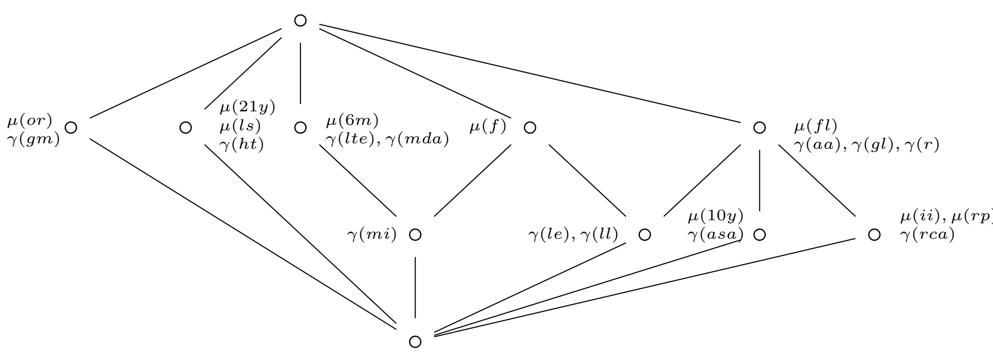

Figure 4: The lattice corresponding to table 3.

In order to bring the principle of transitivity into the picture, note that it is not usually the case that the sanctions as defined in a penal statute figure as applicability conditions/legal grounds in other statutes literally, for whereas the punishments of penal law are usually specific—at least within a reasonable range—about the form, duration and severity of a sanction, legal grounds are typically more general, and, in the case of sanctions, grouped into transgressions that are deemed to be relevantly similar.

To make this more concrete, consider the device of legal debarment, i.e.

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the act of depriving a person of certain legal rights as a consequence of his or her legal history. Common examples include exclusion from public office, suspension, exclusion from various kinds of occupation and/or ineligibility for rendering a public service such as e.g. jury duty. By way of example, consider tables 4 and 5:

	rlt	**sw**	**hp**	**jd**
≥6m	×			×
ic		×	×	×
or				×

Table 4: Proviso-preclusion

Legal provisos:	
≥6m	= more than 6 months imprisonment
ic	= indeterminate custody
or	= official reproof/reprimande
Preclusive effect:	
rlt	= realtor
sw	= social worker
hp	= healthcare professional
jd	= jury duty

Table 5: Legal provisos and preclusions.

Each row in the upper half of table 5 represents a general form of punishment that in this context is to be considered a legal proviso—i.e. it serves to ensure that transgressions falling under it, as specified by penal law, is to have as a legal consequence that there is a capacity in the lower half of table 5 that the offender shall not be allowed to operate in. Suppose provisos and preclusions are correlated as in table 4. Each correlation may be thought of as a point of contact between the penal code from table 3 and some other area of law, for instance a health care professionals act or real estate law.

As to the question of how to render this formally, theorem 4.6 shows that if one adopts basic reusable output as one's logic of norms, then the cumulative effect of tables 3 and 4 can be visualized by taking the union of the two tables, call it G and saturating the resulting relation, call it G^+, before generating the corresponding concept lattice $\mathfrak{B}(G^+)$.

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By definition 4.1, saturating G requires one to determine entailment relationships between extents and intents in $\mathfrak{B}(G)$ first of all. Suppose for the sake of argument that punishments from table 2 are subsumed by the legal provisos from table 5 as follows:

$$10y \vee 21y \ \vdash \geq 6m$$
$$ii \vee rp \ \vdash \ ic$$
$$or \vdash or$$

As examples of relevant entailment relationships in $\mathfrak{B}(G)$ consider the following concepts:

$$(10y^{\lhd}, 10y^{\lhd\rhd}) = (\{asa\}, \{10y\})$$
$$(\geq 6m^{\rhd\lhd}, \geq 6m^{\rhd}) = (\{\geq 6m\}, \{rlt, jd\})$$

Since $10y \ \vdash \geq 6m$ it follows by definition 4.1 that the pairs $(10y, rlt)$, $(10y, jd)$ belong to the saturated context. Repeating the process of deriving new pairs in this way until a fixed point is reached produces the context in table 6:

	6m	**10y**	**21y**	**f**	**fl**	**ls**	**rp**	**ii**	**or**	**rlt**	**sw**	**hp**	**jd**
mi	×			×									
gl					×								
ll				×	×								
r					×								
lte	×					×							
aa					×								
le				×	×								
asa		×			×					∘			∘
rca					×		×	×			∘	∘	∘
nda	×												
ht			×			×				∘			∘
gm									×				×
≥6m										+			+
ic											+	+	+
or													+

Table 6: The saturated context

Here, correlations generated by saturation have been marked with a '∘' instead of a '+'.

This context, table 6 that is, induces the lattice in figure 5 in which, due to saturation, previously implicit information is now manifest. This structure

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is therefore very well suited for querying the cumulative effects of the legal modules in question.

Consider for instance the question “what are the legal consequences of spying on the military, or on the diplomacy, or on the secret services for a hostile and foreign power”. If one takes it that each of these disjuncts imply high treason, then the disjunction as such, call it a, implies high treason. Now, theorem 4.8 says that $\mathbb{E}_{G^+}(a)$ collects legal consequences of a that are all valid according to basic reusable output, and under the assumption that $a \vdash ht$ one has $\mathbb{E}_{G^+}(a) = \{(\{ht\}, \{21y, ls, jd, rlt\})\}$. Thus the legal consequences of spying on the military, or the diplomacy, or the secret services for a hostile and foreign power is $Cn(\{21y, ls, jd, rlt\})$ where Cn is the operation of classical consequence. Thus a person found guilty of espionage will in addition to being liable to imprisonment of up to 21 years, face a number of other sanctions that arise from the interaction of the penal statute with other areas of law. For instance, he or she will not be eligible for jury duty, does not necessarily retain his or her right to vote, and cannot become a realtor—a bit of a surprise at the end there.

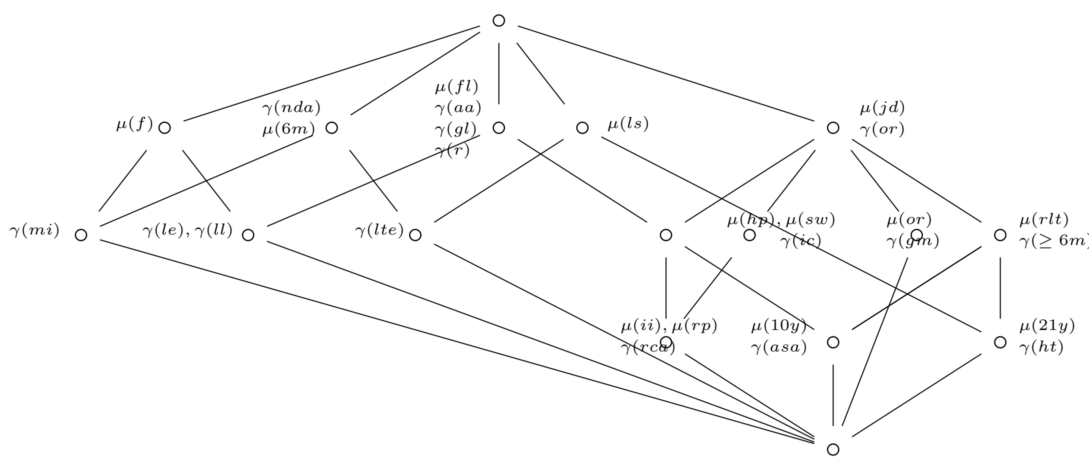

Figure 5: The lattice corresponding to the saturated context

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6. Related work

6.1. The theory of joining systems

In a series of papers going back at least as far as 1999 (thus predating input/output logic) Lars Lindahl and Jan Odelstad have developed an algebraic theory of relations between pre-ordered sets—or, as they prefer to call them, *quasi-orderings*—called the theory of joining systems. The most recent, and most complete presentation of this theory is [9], which also contains references to earlier stages of research.

The theory of joining systems bears some striking resemblances to input/output logic. Philosophically, both idioms were originally motivated by the study of normative systems, and both idioms view norms, not as conditionals capable of being true and false, but rather as logically arbitrary stipulations correlating applicability conditions with states of affairs deemed optimal relative to some end or purpose. Both theories acknowledge the influence of the tradition from Alchourrón and Bulygin in this respect [1, 2].

The theory of joining system itself is not tied to this interpretation, however, but is entirely abstract: the carrier sets of the quasi-orders in question need not be construed concretely as sets of formulae. For present purposes it will be convenient to do so, though. Indeed, for comparison with input/output logic, the case of interest is that where the quasi-orders are boolean algebras generated by the elementary letters of some propositional language.

So interpreted, a joining system is a tuple $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{J} \rangle$ where each $\mathcal{B}_i$ is a boolean algebra, that is, $\mathcal{B}_i = \langle A_i, \leq \rangle$ where A_i is a set of formulae and $\leq$ is the induced partial order. The set of *joinings* $\mathcal{J}$ is a relation from $A_1 \times A_2$ which is constrained in certain ways in order induce logical behaviour. Define the *narrowness* relation $\trianglelefteq$ on $A_1 \times A_2$ by stipulating that $(a_1, b_1) \trianglelefteq (a_2, b_2)$ iff $a_2 \leq a_1$ and $b_1 \leq b_2$. The joining system $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{J} \rangle$ is required to satisfy the following conditions for all $a, b, a_1, a_2, b_1, b_2 \in A_1 \cup A_2$:

1. if $(a_1, b_1) \in \mathcal{J}$ and $(a_1, b_1) \trianglelefteq (a_2, b_2)$, then $(a_2, b_2) \in \mathcal{J}$
2. if $(a, b) \in \mathcal{J}$ for all $a \in A' \subseteq A_1$ then $(\bigvee A', b) \in \mathcal{J}$
3. if $(a, b) \in \mathcal{J}$ for all $b \in A' \subseteq A_2$ then $(a, \bigwedge A') \in \mathcal{J}$

In [9, p. 631] it is conjectured that for this particular case of a joining system we have $\mathcal{J} = out_1(\mathcal{J})$, where out_1 is the operation of simple-minded output as defined in [13].

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Whether this conjecture is indeed true, is a question that is here left open. The interested reader is referred to [22] for a detailed comparison of the two idioms.

Suffice it for present purposes to say that what matters matters for the *informativeness* of a concept lattice is ultimately the syntactic form or shape of the set of axioms for an input/output system. So even if it were the case that a system $C_n(G)$ of of input/output logic could be represented as a set of joinings $\mathcal{J}$, the relationship between $\mathcal{J}$ and the concept lattice induced by G would not thereby necessarily have been determined.

6.2. Related work in input/output logic

Driven by different considerations and objectives, various modifications of the original input/output logic semantics have been proposed. Bochman [3] sees input/output logic primarily as a formalism for reasoning about production systems and causal inference, and furnishes it with a semantics formulated in terms of deductively closed theories called bimodels. Another approach that belongs to the same general field is Goncalves and Alferes [6], who recast input/output logic as an extension of answer set programming, thereby extending it with non-monotonic features.

Some modifications to the input/output idiom have also been proposed by normative systems theorists, who as a group tend to stay closer to the original semantic idiom. Most of these approaches are developed from a classical basis, but uses it differently to achieve different effects than the original input/output semantics. Stolpe [19] considers a weakening of input/output logic that retains input strengthening whilst discarding output weakening in the context of a transitive logic. This involves redefining the input/output semantics in an inductive manner not unlike the inductive reformulation of Reiter default logic given in Makinson [12]. Parent and van der Torre [17] extend the work of [19] in two directions: first the logic is made to support reasoning by cases alias OR, and secondly cumulative transitivity is replaced with a a principle the authors call aggregative cumulative transitivity, which differs from the former principle in keeping tabs on the 'history' of the chaining process. Parent and van der Torre [17] is in turn is used in Sun and van der Torre [21] in studying the interplay between regulative and constitutive rules in a normative system.

As regards approaches to input/output logic that do not take classical logic as the underlying propositional language, we are aware of only one. This is Parent et al. [18] which substitutes intuitionistic logic for classical logic and tracks the changes that must be made to the semantic idiom in order to obtain the same syntactic characterisation of the different input/output systems, up to the meaning of the connectives.

As regards the interface between input/output logic and abstract algebra, there does not seem to be many forerunners to the present paper. The correspondence between the system of basic output and formal concept lattices was first noticed in Stolpe [20], without being developed into a systematic account.

A good introduction to some of the input/output logics currently in existence is [16].

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